ISRAEL JOURNAL OF MATHEMATICS 141 (2004), 61–82

A PURELY INFINITE AH-ALGEBRA AND AN APPLICATION TO AF-EMBEDDABILITY

BY

MIKAEL RØRDAM

Department of Mathematics, University of Southern Denmark
Odense, Campusvej 55, 5230 Odense M, Denmark
e-mail: mikael@imada.sdu.dk
URL: www.imada.sdu.dk/~mikael/

ABSTRACT

We show that there exists a purely infinite AH-algebra. The AH-algebra arises as an inductive limit of C^*-algebras of the form $C_0([0,1), M_k)$ and it absorbs the Cuntz algebra $\mathcal{O}_\infty$ tensorially. Thus one can reach an $\mathcal{O}_\infty$-absorbing C^*-algebra as an inductive limit of the finite and elementary C^*-algebras $C_0([0,1), M_k)$.

As an application we give a new proof of a recent theorem of Ozawa that the cone over any separable exact C^*-algebra is AF-embeddable, and we exhibit a concrete AF-algebra into which this class of C^*-algebras can be embedded.

1. Introduction

Simple C^*-algebras are divided into two disjoint subclasses: those that are stably finite and those that are stably infinite. (A simple C^*-algebra A is stably infinite if $A \otimes \mathcal{K}$ contains an infinite projection, and it is stably finite otherwise.) All simple, stably finite C^*-algebras admit a non-zero quasi-trace, and all exact, simple, stably finite C^*-algebras admit a non-zero trace.

A (possibly non-simple) C^*-algebra A is in [12] defined to be **purely infinite** if no non-zero quotient of A is abelian and if for all positive elements a, b in A, such that b belongs to the closed two-sided ideal generated by a, there is a sequence $\{x_n\}$ of elements in A with $x_n^* a x_n \to b$. Non-simple purely infinite C^*-algebras have been investigated in [12], [13], and [3]. All simple purely infinite C^*-algebras are stably infinite, but the opposite does not hold, cf. [17].

Received June 6, 2002

The condition on a (non-simple) C^*-algebra A, that all projections in $A \otimes \mathcal{K}$ are finite, does not ensure existence of (partially defined) quasi-traces. There are stably projectionless purely infinite C^*-algebras— take for example $C_0(\mathbb{R}) \otimes \mathcal{O}_\infty$, where $\mathcal{O}_\infty$ is the Cuntz algebra generated by a sequence of isometries with pairwise orthogonal range projections—and purely infinite C^*-algebras are traceless.

That stably projectionless purely infinite C^*-algebras can share properties that one would expect are enjoyed only by finite C^*-algebras was demonstrated in a recent paper by Ozawa, [16], in which is it shown that the suspension and the cone over any separable, exact C^*-algebra can be embedded into an AF-algebra. (It seems off hand reasonable to characterize AF-embeddability as a finiteness property.) In particular, $C_0(\mathbb{R}) \otimes \mathcal{O}_\infty$ is AF-embeddable and at the same time purely infinite and traceless. It is surprising that one can embed a traceless C^*-algebra into an AF-algebra, because AF-algebras are well-supplied with traces. If φ: $C_0(\mathbb{R}) \otimes \mathcal{O}_\infty \to A$ is an embedding into an AF-algebra A, then $\mathrm{Im}(\varphi) \cap \mathrm{Dom}(\tau) \subseteq \mathrm{Ker}(\tau)$ for every trace τ on A. This can happen only if the ideal lattice of A has a sub-lattice isomorphic to the interval $[0,1]$ (see Proposition 4.3). In particular A cannot be simple.

Voiculescu's theorem, that the cone and the suspension over any separable C^*-algebra is quasi-diagonal, [19], is a crucial ingredient in Ozawa's proof.

By a construction of Mortensen, [15], there is to each totally ordered, compact, metrizable set T an AH-algebra $\mathcal{A}_T$ with ideal lattice T (cf. Section 2). A C^*-algebra is an AH-algebra, in the sense of Blackadar [1], if it is the inductive limit of a sequence of C^*-algebras each of which is a direct sum of C^*-algebras of the form $M_n(C_0(\Omega)) = C_0(\Omega, M_n)$ (where n and Ω are allowed to vary). We show in Theorem 3.2 (in combination with Proposition 5.2) that the AH-algebra $\mathcal{A}_{[0,1]}$ is purely infinite (and hence traceless)—even in the strong sense that it absorbs $\mathcal{O}_\infty$, i.e., $\mathcal{A}_{[0,1]} \cong \mathcal{A}_{[0,1]} \otimes \mathcal{O}_\infty$— and $\mathcal{A}_{[0,1]}$ is an inductive limit of C^*-algebras of the form $C_0([0,1), M_{2^n})$. We can rephrase this result as follows: Take the smallest class of C^*-algebras, that contains all abelian C^*-algebras and that is closed under direct sums, inductive limits, and stable isomorphism. Then this class contains a purely infinite C^*-algebra (because it contains all AH-algebras).

A word of warning: In the literature, an AH-algebra is often defined to be an inductive limit of direct sums of building blocks of the form $pC(\Omega, M_n)p$, where each Ω is a *compact* Hausdorff space (and p is a projection in $C(\Omega, M_n)$). With this definition, AH-algebras always contain non-zero projections. The algebras we consider, where the building blocs are of the form $C_0(\Omega, M_n)$ for some *locally* *compact* Hausdorff space, should perhaps be called AH_0-algebras to distinguish them from the compact case, but hoping that no confusion will arise, we shall not distinguish between AH- and AH_0-algebras here.

Every AH-algebra is AF-embeddable. Our Theorem 3.2 therefore gives a new proof of Ozawa's result that there are purely infinite—even $\mathcal{O}_\infty$-absorbing—AF-embeddable C^*-algebras. Moreover, just knowing that there exists one AF-embeddable $\mathcal{O}_\infty$-absorbing C^*-algebra, in combination with Kirchberg's theorem that all separable, exact C^*-algebras can be embedded in $\mathcal{O}_\infty$, immediately implies that the cone and the suspension over any separable, exact C^*-algebra is AF-embeddable (Theorem 4.2). This observation yields a new proof of Ozawa's theorem referred to above.

Section 5 contains some results with relevance to the classification program of Elliott. In Section 6 we show that $\mathcal{A}_{[0,1]}$ can be embedded into the AF-algebra $\mathcal{A}_\Omega$, where Ω is the Cantor set, and hence that the cone and the suspension over any separable, exact C^*-algebra can be embedded into this AF-algebra. The ordered K_0-group of $\mathcal{A}_\Omega$ is determined.

ACKNOWLEDGEMENT: I thank Nate Brown for several discussions on quasi-diagonal C^*-algebras and on the possibility of embedding quasi-diagonal C^*-algebras into AF-algebras. I thank Eberhard Kirchberg for suggesting the nice proof of Proposition 4.1, and I thank the referee for suggesting several improvements to the paper.

2. The C^*-algebras $\mathcal{A}_T$

We review in this section results from Mortensen's paper [15] on how to associate a C^*-algebra $\mathcal{A}_T$ with each totally ordered, compact, metrizable set T, so that the ideal lattice of $\mathcal{A}_T$ is order isomorphic to T. Where Mortensen's algebras are inductive limits of C^*-algebras of the form $C_0(T \setminus \{\max T\}, M_{2^n}(\mathcal{O}_2))$, we consider plain matrix algebras M_{2^n} in the place of $M_{2^n}(\mathcal{O}_2)$. It turns out that Mortensen's algebras and those we consider actually are isomorphic when $T = [0,1]$ (see the second paragraph of Section 5).

Any totally ordered set, which is compact and metrizable in its order topology, is order isomorphic to a compact subset of $\mathbb{R}$ (where subsets of $\mathbb{R}$ are given the order structure inherited from $\mathbb{R}$). We shall therefore assume that we are given a compact subset T of $\mathbb{R}$.

Put $t_{\max} = \max T$, $t_{\min} = \min T$, and put $T_0 = T \setminus \{t_{\max}\}$. Choose a sequence $\{t_n\}_{n=1}^\infty$ in T_0 such that the tail $\{t_k, t_{k+1}, t_{k+2}, \dots\}$ is dense in T_0 for every

$k \in \mathbb{N}$. Let $\mathcal{A}_T$ be the inductive limit of the sequence

$$(2.1)\qquad C_0(T_0, M_2) \xrightarrow{\varphi_1} C_0(T_0, M_4) \xrightarrow{\varphi_2} C_0(T_0, M_8) \xrightarrow{\varphi_3} \cdots \longrightarrow \mathcal{A}_T,$$

where

$$(2.2)\qquad \varphi_n(f)(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(\max\{t, t_n\}) \end{pmatrix} = \begin{pmatrix} f(t) & 0 \\ 0 & (f \circ \chi_{t_n})(t) \end{pmatrix},$$

and where for each s in T we let $\chi_s \colon T \to T$ be the continuous function given by $\chi_s(t) = \max\{t, s\}$. The algebra $\mathcal{A}_T$ depends a priori on the choice of the dense sequence $\{t_n\}$. The isomorphism class of $\mathcal{A}_T$ does not depend on this choice when T is the Cantor set (as shown in Section 6) or when T is the interval $[0,1]$ (as will be shown in a forthcoming paper, [14]). It is likely that $\mathcal{A}_T$ is independent on $\{t_n\}$ for arbitrary T.

Put $A_n = C_0(T_0, M_{2^n}) = C_0(T_0) \otimes M_{2^n}$. Let $\varphi_{\infty,n} \colon A_n \to \mathcal{A}_T$ and $\varphi_{m,n} \colon A_n \to A_m$, for $n < m$, denote the inductive limit maps, so that $\mathcal{A}_T$ is the closure of $\bigcup_{n=1}^{\infty} \varphi_{\infty,n}(A_n)$.

Use the identity $\chi_s \circ \chi_t = \chi_{\max\{s,t\}}$ to see that

$$(2.3)\qquad \varphi_{n+k,n}(f) = \begin{pmatrix} f \circ \chi_{\max F_1} & 0 & \cdots & 0 \\ 0 & f \circ \chi_{\max F_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \circ \chi_{\max F_{2^k}} \end{pmatrix}$$

(with the convention $\max \emptyset = t_{\min}$), where $F_1, F_2, \ldots, F_{2^k}$ is an enumeration of the subsets of $\{t_n, t_{n+1}, \ldots, t_{n+k-1}\}$. Note that $\chi_{t_{\min}}$ is the identity map on T.

For each $t \in T$ and for each $n \in \mathbb{N}$ consider the closed ideal

$$(2.4)\qquad I_t^{(n)} \stackrel{\text{def}}{=} \{f \in A_n \mid f(s) = 0 \text{ when } s \geq t\} \cong C_0(T \cap [t_{\min}, t), M_{2^n})$$

of A_n. Observe that $I_{t_{\min}}^{(n)} = \{0\}$, $I_{t_{\max}}^{(n)} = A_n$, and $I_t^{(n)} \subset I_s^{(n)}$ whenever $t < s$ for all $n \in \mathbb{N}$. We have $\varphi_n^{-1}(I_t^{(n+1)}) = I_t^{(n)}$ for all t and for all n, and so

$$(2.5)\qquad I_t \stackrel{\text{def}}{=} \overline{\bigcup_{n=1}^{\infty} \varphi_{\infty,n}(I_t^{(n)})}, \quad t \in T,$$

is a closed two-sided ideal in $\mathcal{A}_T$ such that $I_t^{(n)} = \varphi_{\infty,n}^{-1}(I_t)$. Moreover, $I_{t_{\min}} = \{0\}$, $I_{t_{\max}} = \mathcal{A}_T$, and $I_t \subset I_s$ whenever $s, t \in T$ and $t < s$.

PROPOSITION 2.1 (cf. Mortensen, [15, Theorem 1.2.1]): *Let T be a compact subset of $\mathbb{R}$. Then each closed two-sided ideal in $\mathcal{A}_T$ is equal to I_t for some*

$t \in T$. *It follows that the map* $t \mapsto I_t$ *is an order isomorphism from the ordered set* T *onto the ideal lattice of* $\mathcal{A}_T$.

Proof: Let I be a closed two-sided ideal in $\mathcal{A}_T$. Put $I^{(n)} = \varphi_{\infty,n}^{-1}(I) \lhd C_0(T_0, M_{2^n}) = A_n$, and put

$$T_n = \bigcap_{f \in I^{(n)}} f^{-1}(\{0\}) \subseteq T, \quad n \in \mathbb{N}.$$

Then $I^{(n)}$ is equal to the set of all continuous functions $f\colon T \to M_{2^n}$ that vanish on T_n. It therefore suffices to show that there is t in T such that $T_n = T \cap [t, t_{\max}]$ for all n, cf. (2.4) and (2.5). Now,

$$(2.6) \qquad T_n = T_{n+1} \cup \chi_{t_n}(T_{n+1}) = \bigcup_{F \subseteq X_{n,k}} \chi_{\max F}(T_{n+k}), \quad n, k \in \mathbb{N},$$

where $X_{n,k} = \{t_n, t_{n+1}, \ldots, t_{n+k-1}\}$; because if we let $T'_{n,k}$ denote the right-hand side of (2.6), then for all $f \in C_0(T_0, M_{2^n}) = A_n$,

$$\begin{aligned} f|_{T_n} \equiv 0 &\iff f \in I^{(n)} \iff \varphi_{n+k,n}(f) \in I^{(n+k)} \\ &\iff \forall s \in T_{n+k} : \varphi_{n+k,n}(f)(s) = 0 \\ &\overset{(2.3)}{\iff} \forall F \subseteq X_{n,k} \forall s \in T_{n+k} : f(\chi_{\max F}(s)) = 0 \\ &\iff f|_{T'_{n,k}} \equiv 0. \end{aligned}$$

It follows from (2.6) that $\min T_n \leq \min T_{n+1}$ for all n; and as

$$\min \chi_{t_n}(T_{n+1}) = \max\{t_n, \min T_{n+1}\} \geq \min T_{n+1},$$

we actually have $\min T_n = \min T_{n+1}$ for all n. Let $t \in T$ be the common minimum. Because t belongs to T_{n+k} for all k, we can use (2.6) to conclude that T_n contains the set $\{t_n, t_{n+1}, t_{n+2}, \ldots\} \cap [t, t_{\max}]$; and this set is by assumption dense in $T \cap (t, t_{\max}]$. This proves the desired identity: $T_n = T \cap [t, t_{\max}]$, because T_n is a closed subset of $T \cap [t, t_{\max}]$ and t belongs to T_n. ∎

PROPOSITION 2.2: *$\mathcal{A}_T$ is stable for every compact subset T of $\mathbb{R}$.*

Proof: Let f be a positive element in the dense subset $C_c(T_0, M_{2^n})$ of A_n and let $m > n$ be chosen such that $f(t) = 0$ for all $t \geq t_{m-1}$. Then $f \circ \chi_{\max F} = 0$ for every subset F of $\{t_n, t_{n+1}, \ldots, t_{m-1}\}$ that contains t_{m-1}. In the description of $\varphi_{m,n}(f)$ in (2.3) we see that $f \circ \chi_{\max F_j} = 0$ for at least every other j. We can therefore find a positive function g in $A_m = C_0(T_0, M_{2^m})$ such that $g \perp \varphi_{m,n}(f)$ and $g \sim \varphi_{m,n}(f)$ (the latter in the sense that $x^*x = g$ and $xx^* = \varphi_{m,n}(f)$ for some $x \in A_m$). It follows from [8, Theorem 2.1 and Proposition 2.2] that $\mathcal{A}_T$ is stable. ∎

3. A purely infinite AH-algebra

We show in this section that the C^*-algebra $\mathcal{A}_{[0,1]}$ is traceless and that $\mathcal{B} = \mathcal{A}_{[0,1]} \otimes M_{2^\infty}$ is purely infinite. (In Section 5 it will be shown that $\mathcal{A}_{[0,1]} \cong \mathcal{B}$.)

Following [13, Definition 4.2] we say that an exact C^*-algebra is **traceless** if it admits no non-zero lower semi-continuous trace (whose domain is allowed to be any algebraic ideal of the C^*-algebra). (By restricting to the case of exact C^*-algebras we can avoid talking about quasi-traces; cf. Haagerup [7] and Kirchberg [10].)

If τ is a trace defined on an algebraic ideal $\mathcal{I}$ of a C^*-algebra B, and if I is the closure of $\mathcal{I}$, then $\mathcal{I}$ contains the Pedersen ideal of I. In particular, $(a-\varepsilon)_+$ belongs to $\mathcal{I}$ for every positive element a in I and for every $\varepsilon > 0$. (Here, $(a-\varepsilon)_+ = f_\varepsilon(a)$, where $f_\varepsilon(t) = \max\{t-\varepsilon, 0\}$. Note that $\|a - (a-\varepsilon)_+\| \le \varepsilon$.)

PROPOSITION 3.1: *The C^*-algebra $\mathcal{A}_{[0,1]}$ is traceless.*

Proof: Assume, to reach a contradiction, that τ is a non-zero, lower semi-continuous, positive trace defined on an algebraic ideal $\mathcal{I}$ of $\mathcal{A}_{[0,1]}$, and let I_t be the closure of $\mathcal{I}$, cf. Proposition 2.1. Since τ is non-zero, I_t is non-zero, and hence $t > 0$.

Identify $I_t^{(n)} = \varphi_{\infty,n}^{-1}(I_t)$ with $C_0([0,t), M_{2^n})$. Put $\mathcal{I}^{(n)} = \varphi_{\infty,n}^{-1}(\mathcal{I})$. If x is a positive element in $I_t^{(n)}$ and if $\varepsilon > 0$, then

$$\varphi_{\infty,n}((x-\varepsilon)_+) = (\varphi_{\infty,n}(x) - \varepsilon)_+ \in \mathcal{I},$$

and so $(x-\varepsilon)_+ \in \mathcal{I}^{(n)}$. This shows that $\mathcal{I}^{(n)}$ is a dense ideal in $I_t^{(n)}$, and hence that $\mathcal{I}^{(n)}$ contains $C_c([0,t), M_{2^n})$.

Let τ_n be the trace on $\mathcal{I}^{(n)}$ defined by $\tau_n(f) = \tau(\varphi_{\infty,n}(f))$. We show that

$$\tau_n(f) = \int_0^t \mathrm{Tr}(f(s))\,d\mu_n(s), \quad f \in C_c([0,t), M_{2^n}), \tag{3.1}$$

for some Radon measure μ_n on $[0,t)$ (where Tr denotes the standard unnormalized trace on M_{2^n}). Use Riesz' representation theorem to find a Radon measure μ_n on $[0,t)$ such that $\tau_n(f) = 2^n \int_0^t f(s)\,d\mu_n(s)$ for all f in $C_c([0,t), \mathbb{C}) \subseteq$

$C_c([0,t), M_{2^n})$. Let $E\colon C_c([0,t), M_{2^n}) \to C_c([0,t), \mathbb{C})$ be the conditional expectation given by $E(f)(t) = 2^{-n}\operatorname{Tr}(f(t))$. Then

$$E(f) \in \overline{\mathrm{co}}\{ufu^* \mid u \text{ is a unitary element in } C([0,t], M_{2^n})\}, \tag{3.2}$$

for $f \in C_c([0,t), M_{2^n})$, from which we see that $\tau_n(f) = \tau_n(E(f))$. This proves that (3.1) holds. Because μ_n is a Radon measure, $\mu_n([0,s]) < \infty$ for all $s \in [0,t)$ and for all $n \in \mathbb{N}$.

Let $\{t_n\}_{n=1}^{\infty}$ be the sequence in T used in the definition of $\mathcal{A}_T$. For each n and k in $\mathbb{N}$ we have $\tau_n = \tau_{n+k} \circ \varphi_{n+k,n}$. Set $X_{k,n} = \{t_n, t_{n+1}, \ldots, t_{n+k-1}\}$ and use (2.3) and (3.1) to see that

$$\begin{aligned}\int_0^t \operatorname{Tr}(f(s))d\mu_n(s) &= \tau_n(f) = \tau_{n+k}(\varphi_{n+k,n}(f))\\ &= \int_0^t \operatorname{Tr}(\varphi_{n+k,n}(f)(s))d\mu_{n+k}(s)\\ &= \sum_{F\subseteq X_{k,n}} \int_0^t \operatorname{Tr}((f\circ\chi_{\max(F)})(s))d\mu_{n+k}(s)\\ &= \sum_{F\subseteq X_{k,n}} \int_0^t \operatorname{Tr}(f(s))d(\mu_{n+k}\circ\chi_{\max(F)}^{-1})(s)\end{aligned}$$

for all $f \in C_c([0,t), M_{2^n})$. This entails that

$$\mu_n = \sum_{F\subseteq X_{k,n}} \mu_{n+k}\circ\chi_{\max(F)}^{-1}, \tag{3.3}$$

for all natural numbers n and k.

We prove next that $\mu_n([0,s]) = 0$ for all natural numbers n and for all s in $[0,t)$. Choose r such that $0 < s < r < t$. Put $Y_{k,n} = X_{k,n} \cap [0,s]$ and put $Z_{k,n} = X_{k,n} \cap [0,r]$. Observe that

$$\chi_u^{-1}([0,v]) = \begin{cases} \emptyset, & \text{if } v < u,\\ [0,v], & \text{if } v \geq u,\end{cases} \tag{3.4}$$

whenever $u, v \in [0,1]$. Use (3.3) and (3.4) to obtain

$$\mu_n([0,r]) = \sum_{F\subseteq Z_{k,n}} \mu_{n+k}([0,r]) = 2^{|Z_{k,n}|}\mu_{n+k}([0,r]). \tag{3.5}$$

Use (3.3), (3.4), and (3.5) to see that

$$\begin{aligned}\mu_n([0,s]) &= \sum_{F\subseteq Y_{k,n}} \mu_{n+k}([0,s]) = 2^{|Y_{k,n}|}\mu_{n+k}([0,s])\\ &\leq 2^{|Y_{k,n}|}\mu_{n+k}([0,r]) = 2^{-(|Z_{k,n}|-|Y_{k,n}|)}\mu_n([0,r]).\end{aligned}$$

As

$$\lim_{k\to\infty}(|Z_{k,n}|-|Y_{k,n}|)=\lim_{k\to\infty}|X_{k,n}\cap(s,r]|=\infty$$

(because $\bigcup_{k=n}^{\infty} X_{k,n}=\{t_n,t_{n+1},\ldots\}$ is dense in [0,1)), and as $\mu_n([0,r])<\infty$, we conclude that $\mu_n([0,s])=0$. It follows that $\mu_n([0,t))=0$, whence μ_n and τ_n are zero for all n.

However, if τ is non-zero, then τ_n must be non-zero for some n. To see this, take a positive element e in $\mathcal{I}$ such that $\tau(e)>0$. Because τ is lower semi-continuous there is $\varepsilon>0$ such that $\tau((e-\varepsilon)_+)>0$. Now, $\mathcal{I}^{(n)}$ is dense in $I_t^{(n)}$ and $\bigcup_{n=1}^{\infty}\varphi_{\infty,n}(I_t^{(n)})$ is dense in $I_t\supset\mathcal{I}$. It follows that we can find $n\in\mathbb{N}$ and a positive element f in $\mathcal{I}^{(n)}$ such that $\|\varphi_{\infty,n}(f)-e\|<\varepsilon$. Use for example [13, Lemma 2.2] to find a contraction $d\in A$ such that $d^*\varphi_{\infty,n}(f)d=(e-\varepsilon)_+$. Put $x=\varphi_{\infty,n}(f)^{1/2}d$. Then

$$\begin{aligned}\tau_n(f)=\tau(\varphi_{\infty,n}(f))&\geq\tau(\varphi_{\infty,n}(f)^{1/2}dd^*\varphi_{\infty,n}(f)^{1/2})\\&=\tau(xx^*)=\tau(x^*x)=\tau((e-\varepsilon)_+)>0,\end{aligned}$$

and this shows that τ_n is non-zero. ∎

In the formulation of the main result below, M_{2^∞} denotes the CAR-algebra, or equivalently the UHF-algebra of type 2^∞.

It is shown in [13, Corollary 9.3] that the following three conditions are equivalent for a separable, stable (or unital), nuclear C^*-algebra B:

(1) $B\cong B\otimes\mathcal{O}_\infty$.
(2) B is purely infinite and approximately divisible.
(3) B is traceless and approximately divisible.

The C^*-algebra $\mathcal{O}_\infty$ is the Cuntz algebra generated by a sequence $\{s_n\}_{n=1}^{\infty}$ of isometries with pairwise orthogonal range projections. Pure infiniteness of (non-simple) C^*-algebras was defined in [12] (see also the Introduction). A (possibly non-unital) C^*-algebra B is said to be **approximately divisible** if for each natural number k there is a sequence of unital *-homomorphisms

$$\psi_n\colon M_k\oplus M_{k+1}\to\mathcal{M}(B)$$

such that $\psi_n(x)b-b\psi_n(x)\to 0$ for all $x\in M_k\oplus M_{k+1}$ and for all $b\in B$, cf. [12, Definition 5.5]. The tensor product $A\otimes M_{2^\infty}$ is approximately divisible for any C^*-algebra A.

THEOREM 3.2: *Put $\mathcal{B}=\mathcal{A}_{[0,1]}\otimes M_{2^\infty}$, where $\mathcal{A}_{[0,1]}$ is as defined in (2.1). Then:*

(1) *$\mathcal{B}$ is an inductive limit*

$$C_0([0,1),M_{k_1})\to C_0([0,1),M_{k_2})\to C_0([0,1),M_{k_3})\to\cdots\to\mathcal{B},$$

for some natural numbers $k_1, k_2, k_3, \ldots$. In particular, $\mathcal{B}$ is an AH-algebra.

(2) *$\mathcal{B}$ is traceless, purely infinite, and $\mathcal{B} \cong \mathcal{B} \otimes \mathcal{O}_\infty$.*

It is shown in Proposition 5.2 below that $\mathcal{A}_{[0,1]} \cong \mathcal{B}$. We stress that this fact will not be used in the proof of Theorem 4.2 below.

Proof: Part (1) follows immediately from the construction of $\mathcal{A}_{[0,1]}$ and from the fact that M_{2^∞} is an inductive limit of matrix algebras.

(2) The property of being traceless is preserved after tensoring with M_{2^∞}, so $\mathcal{B}$ is traceless by Proposition 3.1. As remarked above, $\mathcal{B}$ is approximately divisible, $\mathcal{A}_{[0,1]}$ and hence $\mathcal{B}$ are stable by Proposition 2.2, and as $\mathcal{B}$ is also nuclear and separable it follows from [13, Corollary 9.3] (quoted above) that $\mathcal{B}$ is purely infinite and $\mathcal{O}_\infty$-absorbing. ∎

The C^*-algebra $\mathcal{B}$ is stably projectionless, and, in fact, every purely infinite AH-algebra is (stably) projectionless. Indeed, any projection in an AH-algebra is finite (in the sense of Murray and von Neumann), and any non-zero projection in a purely infinite C^*-algebra is (properly) infinite, cf. [12, Theorem 4.16].

It is impossible to find a *simple* purely infinite AH-algebra, because all simple purely infinite C^*-algebras contain properly infinite projections.

4. An application to AF-embeddability

We show here how Theorem 3.2 leads to a new proof of the recent theorem of Ozawa that the cone and the suspension over any exact separable C^*-algebra are AF-embeddable, [16].

It is well-known that any ASH-algebra, hence any AH-algebra, and hence the C^*-algebras $\mathcal{A}_{[0,1]}$ and $\mathcal{B}$ from Theorem 3.2 are AF-embeddable. For the convenience of the reader we include a proof of this fact—the proof we present is due to Kirchberg. (An ASH-algebra is a C^*-algebra that arises as the inductive limit of a sequence of C^*-algebras each of which is a finite direct sum of basic building blocks: sub-C^*-algebras of $M_n(C_0(\Omega))$—where n and Ω are allowed to vary.)

An embedding of $\mathcal{A}_{[0,1]}$ into an explicit AF-algebra is given in Section 6.

PROPOSITION 4.1 (Folklore): *Every ASH-algebra admits a faithful embedding into an AF-algebra.*

Proof: Note first that if A is a sub-C^*-algebra of $M_n(C_0(\Omega))$, then its enveloping von Neumann algebra A^{**} is isomorphic to $\bigoplus_{k=1}^n M_k(\mathcal{C}_k)$ for some (possibly trivial) abelian von Neumann algebras $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$. If $\mathcal{C}$ is an abelian von

Neumann algebra and if D is a separable sub-C^*-algebra of $M_k(\mathcal{C})$, then there is a (separable) sub-C^*-algebra D_1 of $M_k(\mathcal{C})$ that contains D and such that $D_1 \cong M_k(C(X))$, where X is a compact Hausdorff space of dimension zero. In particular, D_1 is an AF-algebra.

To see this, let D_0 be the separable C^*-algebra generated by D and the matrix units of $M_k \subseteq M_k(\mathcal{C})$. Then $D_0 = M_k(\mathcal{D}_0)$ for some separable sub-C^*-algebra $\mathcal{D}_0$ of $\mathcal{C}$. Any separable sub-C^*-algebra of a (possibly non-separable) C^*-algebra of real rank zero is contained in a separable sub-C^*-algebra of real rank zero. (This is obtained by successively adding projections from the bigger C^*-algebra.) Hence $\mathcal{D}_0$ is contained in a separable real rank zero sub-C^*-algebra $\mathcal{D}_1$ of $\mathcal{C}$. It follows from [4] that $\mathcal{D}_1 \cong C(X)$ for some zero-dimensional compact Hausdorff space X. Hence $D_1 = M_k(\mathcal{D}_1)$ is as desired.

Assume now that B is an ASH-algebra, so that it is an inductive limit

$$B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} B_3 \xrightarrow{\psi_3} \cdots \longrightarrow B,$$

where each B_n is a finite direct sum of sub-C^*-algebras of $M_m(C_0(\Omega))$. Passing to the bi-dual we get a sequence of finite von Neumann algebras

$$B_1^{**} \xrightarrow{\psi_1^{**}} B_2^{**} \xrightarrow{\psi_2^{**}} B_3^{**} \xrightarrow{\psi_3^{**}} \cdots .$$

Use the observation from the first paragraph (now applied to direct sums of basic building blocks) to find an AF-algebra D_1 such that $B_1 \subseteq D_1 \subseteq B_1^{**}$. Use the observation again to find an AF-algebra D_2 such that $C^*(\psi_1^{**}(D_1), B_2) \subseteq D_2 \subseteq B_2^{**}$. Continue in this way and find, at the nth stage, an AF-algebra D_n such that $C^*(\psi_{n-1}^{**}(D_{n-1}), B_n) \subseteq D_n \subseteq B_n^{**}$. It then follows that the inductive limit D of

$$D_1 \xrightarrow{\psi_1^{**}} D_2 \xrightarrow{\psi_2^{**}} D_3 \xrightarrow{\psi_3^{**}} \cdots \longrightarrow D$$

is an AF-algebra that contains B. ∎

THEOREM 4.2 (Ozawa): *The cone $CA = C_0([0,1), A)$ over any separable exact C^*-algebra A admits a faithful embedding into an AF-algebra.*

Proof: By a renowned theorem of Kirchberg any separable exact C^*-algebra can be embedded into the Cuntz algebra $\mathcal{O}_2$ (see [11]), and hence into $\mathcal{O}_\infty$ (the latter because $\mathcal{O}_2$ can be embedded—non-unitally—into $\mathcal{O}_\infty$). It therefore suffices to show that $C\mathcal{O}_\infty = C_0([0,1)) \otimes \mathcal{O}_\infty$ is AF-embeddable. It is clear from the construction of $\mathcal{B}$ in Theorem 3.2 that $C_0([0,1))$ admits an embedding into the C^*-algebra $\mathcal{B}$. (Actually, one can embed $C_0([0,1))$ into any C^*-algebra that absorbs $\mathcal{O}_\infty$.) As $\mathcal{B} \cong \mathcal{B} \otimes \mathcal{O}_\infty$, we can embed $C\mathcal{O}_\infty$ into $\mathcal{B}$. Now, $\mathcal{B}$ is an AH-algebra and therefore AF-embeddable, cf. Proposition 4.1, so $C\mathcal{O}_\infty$ is AF-embeddable. ∎

Ozawa used his theorem in combination with a result of Spielberg to conclude that the class of AF-embeddable C^*-algebras is closed under homotopy invariance, and even more: If A is AF-embeddable and B is homotopically dominated by A, then B is AF-embeddable.

The suspension $SA = C_0((0,1), A)$ is a sub-C^*-algebra of CA, and so it follows from Theorem 4.2 that also the suspension over any separable exact C^*-algebra is AF-embeddable.

No simple AF-algebra contains a purely infinite sub-C^*-algebra. In fact, any AF-algebra, that has a purely infinite sub-C^*-algebra, must have uncountably many ideals:

PROPOSITION 4.3: *Suppose that $\varphi\colon A \to B$ is an embedding of a purely infinite C^*-algebra A into an AF-algebra B. Let a be a non-zero positive element in $\mathrm{Im}(\varphi)$. For each t in $[0, \|a\|]$ let I_t be the closed two-sided ideal in B generated by $(a-t)_+$. Then the map $t \mapsto I_{\|a\|-t}$ defines an injective order embedding of the interval $[0, \|a\|]$ into the ideal lattice of B.*

Proof: Since A is traceless (being purely infinite, cf. [12]), $\mathrm{Im}(\varphi) \cap \mathrm{Dom}(\tau) \subseteq \mathrm{Ker}(\tau)$ for every trace τ on B.

Let $0 \le t < s \le \|a\|$ be given. We show that I_s is strictly contained in I_t. Find a projection p in $\overline{(a-t)_+ B (a-t)_+}$ such that $\|(a-t)_+ - p(a-t)_+ p\| < s - t$. There is a trace τ, defined on the algebraic ideal in B generated by p, with $\tau(p) = 1$. We claim that

$$I_s \subseteq \mathrm{Ker}(\tau) \subset \mathrm{Dom}(\tau) \subseteq I_t,$$

and this will prove the proposition. To see the first inclusion, there is d in B such that $(a-s)_+ = d^* p (a-t)_+ p d$ (use for example [13, Lemma 2.2 and (2.1)]). Therefore $(a-s)_+$ belongs to the algebraic ideal generated by p, whence $(a-s)_+ \in \mathrm{Im}(\varphi) \cap \mathrm{Dom}(\tau) \subseteq \mathrm{Ker}(\tau)$. This entails that I_s is contained in the kernel of τ.

The strict middle inclusion holds because $0 < \tau(p) < \infty$. The last inclusion holds because p belongs to $\overline{(a-t)_+ B (a-t)_+} \subseteq I_t$. ∎

It follows from Proposition 5.1 below that no AF-algebra can have ideal lattice isomorphic to $[0,1]$, and so the order embedding from Proposition 4.3 can never be surjective. In Section 6 we show that one can embed a (stably projectionless) purely infinite C^*-algebra into the AF-algebra $\mathcal{A}_\Omega$, where Ω is the Cantor set. The ideal lattice of $\mathcal{A}_\Omega$ is the totally ordered and totally disconnected set Ω.

5. Further properties of the algebras $\mathcal{A}_T$

Nuclear separable C^*-algebras that absorb $\mathcal{O}_\infty$ have been classified by Kirchberg in terms of an ideal preserving version of Kasparov's KK-theory, see [9]. It is not easy to decide when two such C^*-algebras with the same primitive ideal space are KK-equivalent in this sense. There is, however, a particularly well understood special case: If A and B are nuclear, separable, stable C^*-algebras that absorb the Cuntz algebra $\mathcal{O}_2$, then A is isomorphic to B if and only if A and B have homeomorphic primitive ideal spaces (cf. Kirchberg, [9]).

We show in this section that $\mathcal{A}_{[0,1]} \cong \mathcal{A}_{[0,1]} \otimes \mathcal{O}_\infty$ and that $\mathcal{A}_{[0,1]}$ is isomorphic to the C^*-algebra $\mathcal{B}$ from Theorem 3.2. It is shown in a forthcoming paper, [14], that $\mathcal{A}_{[0,1]} \cong \mathcal{A}_{[0,1]} \otimes \mathcal{O}_2$ (using an observartion that $\mathcal{A}_{[0,1]}$ is zero homotopic in an ideal-system preserving way, i.e., there is a *-homomorphism $\Psi\colon \mathcal{A}_{[0,1]} \to C_0([0,1), \mathcal{A}_{[0,1]})$ such that $\mathrm{ev}_0 \circ \Psi = \mathrm{id}_{\mathcal{A}_{[0,1]}}$ and $\Psi(J) \subseteq C_0([0,1), J)$ for every closed two-sided ideal J in $\mathcal{A}_{[0,1]}$). Thus it follows from Kirchberg's theorem that $\mathcal{A}_{[0,1]}$ is the *unique* separable, nuclear, stable, $\mathcal{O}_2$-absorbing C^*-algebra whose ideal lattice is (order isomorphic to) $[0,1]$. It seems likely (but remains open) that any separable, nuclear, traceless C^*-algebra with ideal lattice isomorphic to $[0,1]$ must absorb $\mathcal{O}_2$ and hence be isomorphic to $\mathcal{A}_{[0,1]}$.

Not all nuclear, separable C^*-algebras, whose ideal lattice is isomorphic to $[0,1]$, are purely infinite (or traceless) as shown in Proposition 5.4 below.

We derive below a couple of facts about C^*-algebras that have ideal lattice isomorphic to $[0,1]$:

PROPOSITION 5.1: *Let D be a C^*-algebra with ideal lattice order isomorphic to $[0,1]$. Then D is stably projectionless. If D moreover is purely infinite and separable, then D is necessarily stable.*

Proof: Since D and $D \otimes \mathcal{K}$ have the same ideal lattice it suffices to show that D contains no non-zero projections. Let $\{I_t \mid t \in [0,1]\}$ be the ideal lattice of D (such that $I_t \subset I_s$ whenever $t < s$). Suppose, to reach a contradiction, that D contains a non-zero projection e. Let I_s be the ideal in D generated by e. The ideal lattice of the unital C^*-algebra eDe is then $\{eI_te \mid t \in [0,s]\}$ and $eI_te \subset eI_re$ whenever $0 \le t < r \le s$. This is in contradiction with the well-known fact that any unital C^*-algebra has a maximal proper ideal.

Suppose now that D is purely infinite and separable. To show that D is stable it suffices to show that D has no (non-zero) unital quotient, cf. [12, Theorem 4.24]. The ideal lattice of an arbitrary quotient D/I_s of D is equal to $\{I_t/I_s \mid t \in [s,1]\}$, and this lattice is order isomorphic to the interval $[0,1]$ (provided that $I_s \neq I_1 = D$). It therefore follows from the first part of the proposition that D/I_s has no non-zero projection and D/I_s is therefore in particular non-unital. ∎

PROPOSITION 5.2: *$\mathcal{A}_T \cong \mathcal{A}_T \otimes M_{2^\infty} \otimes \mathcal{K}$ for every compact subset T of $\mathbb{R}$.*

Proof: It was shown in Proposition 2.2 that $\mathcal{A}_T$ is stable. We proceed to show that $\mathcal{A}_T$ is isomorphic to $\mathcal{A}_T \otimes M_{2^\infty}$. Recall that $A_n = C_0(T_0, M_{2^n})$, put $\widetilde{A}_n = C(T, M_{2^n})$, and consider the commutative diagram:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\varphi_1} & A_2 & \xrightarrow{\varphi_2} & A_3 & \xrightarrow{\varphi_3} & \cdots & \longrightarrow & \mathcal{A}_T \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \widetilde{A}_1 & \xrightarrow[\widetilde{\varphi}_1]{} & \widetilde{A}_2 & \xrightarrow[\widetilde{\varphi}_2]{} & \widetilde{A}_3 & \xrightarrow[\widetilde{\varphi}_3]{} & \cdots & \longrightarrow & \widetilde{\mathcal{A}}, \end{array}$$

where φ_n is as defined in (2.2), and where $\widetilde{\varphi}_n\colon \widetilde{A}_n \to \widetilde{A}_{n+1}$ is defined using the same recipe as in (2.2). The inductive limit C^*-algebra $\widetilde{\mathcal{A}}$ is unital, each A_n is an ideal in $\widetilde{A}_n$, and $\mathcal{A}_T$ is (isomorphic to) an ideal in $\widetilde{\mathcal{A}}$.

We show that $\widetilde{\mathcal{A}} \cong \widetilde{\mathcal{A}} \otimes M_{2^\infty}$. This will imply that $\mathcal{A}_T$ is isomorphic to an ideal of $\widetilde{\mathcal{A}} \otimes M_{2^\infty}$. Each ideal in $\widetilde{\mathcal{A}} \otimes M_{2^\infty}$ is of the form $I \otimes M_{2^\infty}$ for some ideal I in $\widetilde{\mathcal{A}}$. As $M_{2^\infty} \cong M_{2^\infty} \otimes M_{2^\infty}$ it will follow that $\mathcal{A}_T \cong \mathcal{A}_T \otimes M_{2^\infty}$.

By [2, Proposition 2.12] (and its proof), to prove that $\widetilde{\mathcal{A}} \cong \widetilde{\mathcal{A}} \otimes M_{2^\infty}$ it suffices to show that for each finite subset G of $\widetilde{\mathcal{A}}$ and for each $\varepsilon > 0$ there is a unital *-homomorphism $\lambda\colon M_2 \to \widetilde{\mathcal{A}}$ such that $\|\lambda(x)g - g\lambda(x)\| \leq \varepsilon\|x\|$ for all $x \in M_2$ and for all $g \in G$. We may assume that G is contained in $\widetilde{\varphi}_{\infty,n}(\widetilde{A}_n)$ for some natural number n. Put $H = \widetilde{\varphi}_{\infty,n}^{-1}(G)$. It now suffices to find a natural number k and a unital *-homomorphism $\lambda\colon M_2 \to \widetilde{A}_{n+k}$ such that

$$(5.1) \qquad \|\lambda(x)\widetilde{\varphi}_{n+k,n}(h) - \widetilde{\varphi}_{n+k,n}(h)\lambda(x)\| \leq \varepsilon\|x\|, \quad x \in M_2, \quad h \in H.$$

Put $t_{\min} = \min T$, and find $\delta > 0$ such that $\|h(t) - h(t_{\min})\| \leq \varepsilon/2$ for all h in H and for all t in T with $|t - t_{\min}| < \delta$. Let $\{t_n\}$ be the dense sequence in T_0 used in the definition of $\mathcal{A}_T$. Find $m \geq n$ such that $|t_m - t_{\min}| < \delta$. Put $k = m + 1 - n$, and organize the elements in $X = \{t_n, t_{n+1}, \ldots, t_{m+1}\}$ in increasing order and relabel the elements by $s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_k$. Let $F_1, F_2, \ldots, F_{2^k}$ be the subsets of X ordered such that $F_1 = \emptyset$ and

$$\max F_2 = s_1, \max F_3 = \max F_4 = s_2, \ldots, \max F_{2^{k-1}+1} = \cdots = \max F_{2^k} = s_k.$$

Then $|s_1 - t_{\min}| < \delta$, and so $\|h \circ \chi_{\max F_1} - h \circ \chi_{\max F_2}\| \leq \varepsilon$ for $h \in H$ (we use the convention $\max \emptyset = t_{\min}$); and $h \circ \chi_{\max F_{2j-1}} = h \circ \chi_{\max F_{2j}}$ when $j \geq 2$ for all h.

We shall use the picture of $\varphi_{n+k,n}$ given in (2.3), which is valid also for $\widetilde{\varphi}_{n+k,n}$. However, since the sets $F_1, F_2, \ldots, F_k$ possibly have been permuted, $\varphi_{n+k,n}$ and the expression in (2.3) agree only up to unitary equivalence. Let $\lambda\colon M_2 \to \widetilde{A}_{n+k}$ be the unital *-homomorphism given by $\lambda(x) = \operatorname{diag}(x, x, \ldots, x)$ (with 2^{k-1} copies of x). Use (2.3) and the estimate

$$\begin{aligned} &\left\| x \begin{pmatrix} h \circ \chi_{\max F_{2j-1}} & 0 \\ 0 & h \circ \chi_{\max F_{2j}} \end{pmatrix} - \begin{pmatrix} h \circ \chi_{\max F_{2j-1}} & 0 \\ 0 & h \circ \chi_{\max F_{2j}} \end{pmatrix} x \right\| \\ \leq & \|x\| \|h \circ \chi_{\max F_{2j-1}} - h \circ \chi_{\max F_{2j}}\| \leq \varepsilon \|x\|, \end{aligned}$$

that holds for $j = 1, 2, \ldots, 2^{k-1}$, for $h \in H$, and for all $x \in M_2(\mathbb{C}) \subseteq C(T, M_2)$, to conclude that (5.1) holds, and hence that $\widetilde{A} \cong \widetilde{A} \otimes M_{2^\infty}$. ∎

Proposition 5.2 together with Theorem 3.2 yield:

Corollary 5.3: *The C^*-algebra $\mathcal{A}_{[0,1]}$ is purely infinite and*

$$\mathcal{A}_{[0,1]} \cong \mathcal{A}_{[0,1]} \otimes \mathcal{O}_\infty.$$

We conclude this section by showing that the tracelessness of the C^*-algebras $\mathcal{A}_{[0,1]}$ (established in Proposition 3.1) is not a consequence of its ideal lattice being isomorphic to $[0, 1]$.

Proposition 5.4: *Let $\{l_n\}_{n=1}^\infty$ be a sequence of positive integers, and let $\{t_n\}_{n=1}^\infty$ be a dense sequence in $[0, 1)$. Put $k_1 = 1$, put $k_{n+1} = (l_n + 1)k_n$ for $n \geq 1$, and put $D_n = C_0([0, 1), M_{k_n})$. Let $\mathcal{D}$ be the inductive limit of the sequence*

$$D_1 \xrightarrow{\psi_1} D_1 \xrightarrow{\psi_2} D_2 \xrightarrow{\psi_3} \cdots \longrightarrow \mathcal{D},$$

where $\psi_n(f) = \operatorname{diag}(f, f, \ldots, f, f \circ \chi_{t_n})$ (with l_n copies of f), and where the map $\chi_{t_n}\colon [0, 1] \to [0, 1]$ as before is given by $\chi_{t_n}(s) = \max\{s, t_n\}$.

It follows that the ideal lattice of $\mathcal{D}$ is isomorphic to the interval $[0, 1]$. Moreover, if $\prod_{n=1}^\infty l_n/(l_n + 1) > 0$, then $\mathcal{D}$ has a non-zero bounded trace, in which case $\mathcal{D}$ is not stable and not purely infinite.

Proof: An obvious modification of the proof of Proposition 2.1 shows that the ideal lattice of $\mathcal{D}$ is isomorphic to $[0,1]$. As in the proof of Proposition 5.2 we construct a unital C^*-algebra $\widetilde{\mathcal{D}}$, in which $\mathcal{D}$ is a closed two-sided ideal, by letting $\widetilde{\mathcal{D}}$ be the inductive limit of the sequence

$$\begin{array}{ccccccccc}
D_1 & \xrightarrow{\psi_1} & D_2 & \xrightarrow{\psi_2} & D_3 & \xrightarrow{\psi_3} & \cdots & \longrightarrow & \mathcal{D} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\widetilde{D}_1 & \xrightarrow[\widetilde{\psi}_1]{} & \widetilde{D}_2 & \xrightarrow[\widetilde{\psi}_2]{} & \widetilde{D}_3 & \xrightarrow[\widetilde{\psi}_3]{} & \cdots & \longrightarrow & \widetilde{\mathcal{D}},
\end{array}$$

where $\widetilde{D}_n = C([0,1], M_{k_n})$ and $\widetilde{\psi}_n(f) = \mathrm{diag}(f, \ldots, f, f \circ \chi_{t_n})$. Remark that

$$\widetilde{\mathcal{D}}/\mathcal{D} \cong \varinjlim \widetilde{D}_n/D_n \cong \varinjlim M_{k_n}$$

is a UHF-algebra. If τ is a tracial state on $\widetilde{\mathcal{D}}$ that vanishes on $\mathcal{D}$, then τ is the composition of the quotient mapping $\widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}/\mathcal{D}$ and the unique tracial state on the UHF-algebra $\widetilde{\mathcal{D}}/\mathcal{D}$. It follows that there is only one tracial state τ on $\widetilde{\mathcal{D}}$ that vanishes on $\mathcal{D}$.

Suppose now that $\prod_{n=1}^{\infty} l_n/(l_n+1) > 0$. It then follows, as in the construction of Goodearl in [6], that the simplex of tracial states on $\widetilde{\mathcal{D}}$ is homeomorphic to the simplex of probability measures on $[0,1]$ and hence that $\widetilde{\mathcal{D}}$ has a tracial state that does not vanish on $\mathcal{D}$. The restriction of this trace to $\mathcal{D}$ is then the desired non-zero bounded trace. (Goodearl constructs simple C^*-algebras; and where $f \circ \chi_{t_n}$ appears in our connecting map $\widetilde{\psi}_n$, Goodearl uses a point evaluation, i.e., the constant function $t \mapsto f(t_n)$. Goodearl's proof can nonetheless and without changes be applied in our situation.) ∎

6. An embedding into a concrete AF-algebra

Let T be a compact subset of $\mathbb{R}$ and set $T_0 = T \backslash \{\max T\}$. Then $C_0(T_0, M_{2^n})$ is an AF-algebra if and only if T is totally disconnected. It follows that the C^*-algebra $\mathcal{A}_T$ (defined in (2.1)) is an AF-algebra whenever T is totally disconnected. Let Ω denote the Cantor set (realized as the "middle third" subset of [0,1], and with the total order it inherits from its embedding in $\mathbb{R}$). Actually any totally disconnected, compact subset of $\mathbb{R}$ with no isolated points is order isomorphic to Ω.

We show here that the AF-algebra from Theorem 4.2, into which the cone over any separable exact C^*-algebra can be embedded, can be chosen to be $\mathcal{A}_\Omega$.

The ideal lattice of $\mathcal{A}_\Omega$ is order isomorphic to Ω (by Proposition 2.1). In the light of Proposition 4.3 and by the fact that the ideal lattice of an AF-algebra is totally disconnected (in an appropriate sense) the AF-algebra $\mathcal{A}_\Omega$ has the least complicated ideal lattice among AF-algebras that admit embeddings of (stably projectionless) purely infinite C^*-algebras.

We begin by proving a general result on when $\mathcal{A}_S$ can be embedded into $\mathcal{A}_T$:

PROPOSITION 6.1: *Let S and T be compact subsets of $\mathbb{R}$. Set $T_0 = T \setminus \{\max T\}$ and $S_0 = S \setminus \{\max S\}$. Suppose there is a continuous, increasing, surjective function $\lambda\colon T \to S$ such that $\lambda(T_0) = S_0$. Let $\{t_n\}_{n=1}^\infty$ be a sequence in T_0 such that $\{t_n\}_{n=k}^\infty$ is dense in T_0 for every k, and put $s_n = \lambda(t_n)$. Then $\{s_n\}_{n=k}^\infty$ is dense in S_0 for every k, and there is an injective *-homomorphism $\lambda^\sharp\colon \mathcal{A}_S \to \mathcal{A}_T$, when $\mathcal{A}_T$ and $\mathcal{A}_S$ are inductive limits as in (2.1) with respect to the sequences $\{t_n\}_{n=1}^\infty$ and $\{s_n\}_{n=1}^\infty$, respectively. If λ moreover is injective, then $\lambda^\sharp$ is an isomorphism.*

Proof: There is a commutative diagram:

$$
\begin{array}{ccccccccc}
C_0(S_0,M_2) & \xrightarrow{\varphi_1} & C_0(S_0,M_4) & \xrightarrow{\varphi_2} & C_0(S_0,M_8) & \xrightarrow{\varphi_3} & \cdots & \longrightarrow & \mathcal{A}_S \\
\Big\downarrow{\scriptstyle\widehat{\lambda}} & & \Big\downarrow{\scriptstyle\widehat{\lambda}} & & \Big\downarrow{\scriptstyle\widehat{\lambda}} & & & & \Big\downarrow{\scriptstyle\lambda^\sharp} \\
C_0(T_0,M_2) & \xrightarrow[\psi_1]{} & C_0(T_0,M_4) & \xrightarrow[\psi_2]{} & C_0(T_0,M_8) & \xrightarrow[\psi_3]{} & \cdots & \longrightarrow & \mathcal{A}_T
\end{array}
\tag{6.1}
$$

where $\widehat{\lambda}(f) = f \circ \lambda$, and where

$$
\varphi_n(f) = \begin{pmatrix} f & 0 \\ 0 & f \circ \chi_{s_n} \end{pmatrix}, \quad \psi_n(f) = \begin{pmatrix} f & 0 \\ 0 & f \circ \chi_{t_n} \end{pmatrix},
\tag{6.2}
$$

cf. (2.2). Note that $\lambda(t_{\max}) = s_{\max}$ (because λ is surjective), and so $\widehat{\lambda}(f)(t_{\max}) = f(\lambda(t_{\max})) = f(s_{\max}) = 0$. To see that the diagram (6.1) indeed is commutative we must check that $\widehat{\lambda} \circ \varphi_n = \psi_n \circ \widehat{\lambda}$ for all n. By (6.2),

$$
(\widehat{\lambda} \circ \varphi_n)(f) = \begin{pmatrix} f \circ \lambda & 0 \\ 0 & f \circ \chi_{s_n} \circ \lambda \end{pmatrix}, \quad (\psi_n \circ \widehat{\lambda})(f) = \begin{pmatrix} f \circ \lambda & 0 \\ 0 & f \circ \lambda \circ \chi_{t_n} \end{pmatrix},
$$

for all $f \in C_0(S_0, M_{2^n})$, so it suffices to check that $\chi_{s_n} \circ \lambda = \lambda \circ \chi_{t_n}$. But

$$
\begin{aligned}
(\chi_{s_n} \circ \lambda)(x) = \max\{\lambda(x), s_n\} = \max\{\lambda(x), \lambda(t_n)\} &= \lambda(\max\{x, t_n\}) \\
&= (\lambda \circ \chi_{t_n})(x),
\end{aligned}
$$

where the third equality holds because λ is increasing.

Each map $\widehat{\lambda}$ in the diagram (6.1) is injective (because λ is surjective), so the *-homomorphism $\lambda^\sharp\colon \mathcal{A}_S \to \mathcal{A}_T$ induced by the diagram is injective.

If λ also is injective, then each map $\widehat{\lambda}$ in (6.1) is an isomorphism in which case $\lambda^\sharp$ is an isomorphism. ∎

Combine (the proof of) Theorem 4.2 with Proposition 5.2 to obtain:

PROPOSITION 6.2: *The cone and the suspension over any separable exact C^*-algebra admits an embedding into the AH-algebra $\mathcal{A}_{[0,1]}$.*

LEMMA 6.3: *There is a continuous, increasing, surjective map $\lambda\colon \Omega \to [0,1]$ that maps $[0,1)$ into Ω_0, where Ω is the Cantor set and where $\Omega_0 = \Omega \setminus \{1\}$.*

Proof: Each x in Ω can be written $x = \sum_{n \in F} 2\cdot 3^{-n}$ for a unique subset F of $\mathbb{N}$. We can therefore define λ by

$$\lambda\Big(\sum_{n\in F} 2\cdot 3^{-n}\Big) = \sum_{n\in F} 2^{-n}, \quad F \subseteq \mathbb{N}.$$

It is straightforward to check that λ has the desired properties. ∎

COROLLARY 6.4: *The cone and the suspension over any separable exact C^*-algebra admits an embedding into the AF-algebra $\mathcal{A}_\Omega$.*

Proof: It follows from Proposition 6.1 and Lemma 6.3 that $\mathcal{A}_{[0,1]}$ can be embedded into $\mathcal{A}_\Omega$. The corollary is now an immediate consequence of Proposition 6.2. ∎

By a renowned theorem of Elliott, [5], the ordered K_0-group is a complete invariant for the stable isomorphism class of an AF-algebra. We shall therefore go to some length to calculate the ordered group $K_0(\mathcal{A}_\Omega)$.

As $K_0(\mathcal{A}_\Omega)$ does not depend on the choice of dense sequence $\{t_n\}_{n=1}^\infty$ used in the inductive limit description of $\mathcal{A}_\Omega$, (2.1), it follows in particular from Proposition 6.5 below that the isomorphism class of $\mathcal{A}_\Omega$ is independent of this sequence.

The Cantor set Ω is realized as the "middle-third" subset of $[0,1]$ (so that $0 = \min \Omega$ and $1 = \max \Omega$). Consider the countable abelian group $G = C_0(\Omega_0, \mathbb{Z}[\frac{1}{2}])$ where the composition is addition, and where the group of Dyadic rationals $\mathbb{Z}[\frac{1}{2}]$ is given the discrete topology. Equip G with the lexicographic order, whereby $f \in G^+$ if and only if either $f = 0$ or $f(t_0) > 0$ for $t_0 = \sup\{t \in \Omega \mid f(t) \neq 0\}$. (The set $\{t \in \Omega \mid f(t) \neq 0\}$ is clopen because $\mathbb{Z}[\frac{1}{2}]$ is discrete, and so $f(t_0) \neq 0$.)

It is easily checked that (G, G^+) is a totally ordered abelian group, and hence a dimension group.

PROPOSITION 6.5: *$K_0(\mathcal{A}_\Omega)$ is order isomorphic to the group $C_0(\Omega_0, \mathbb{Z}[\frac{1}{2}])$ equipped with the lexicographic ordering.*

Proof: Let $\{t_n\}_{n=1}^\infty$ be any sequence in $\Omega_0 = \Omega\backslash\{1\}$ such that $\{t_k, t_{k+1}, t_{k+2}, \dots\}$ is dense in Ω_0 for all k. Write $\mathcal{A}_\Omega$ as an inductive limit with connecting maps φ_n as in (2.1).

By continuity of K_0 and because $K_0(C_0(\Omega_0, M_{2^n})) \cong C_0(\Omega_0, \mathbb{Z})$ (as ordered abelian groups) (see e.g. [18, Exercise 3.4]), the ordered abelian group $K_0(\mathcal{A}_\Omega)$ is the inductive limit of the sequence

$$C_0(\Omega_0,\mathbb{Z}) \xrightarrow{\alpha_1} C_0(\Omega_0,\mathbb{Z}) \xrightarrow{\alpha_2} C_0(\Omega_0,\mathbb{Z}) \xrightarrow{\alpha_3} \cdots \longrightarrow K_0(\mathcal{A}_\Omega),$$

where $\alpha_n(f) = K_0(\varphi_n)(f) = f + f \circ \chi_{t_n}$.

Choose for each $n \in \mathbb{N}$ a partition $\{A_1^{(n)}, A_2^{(n)}, \dots, A_{2^n}^{(n)}\}$ of Ω into clopen intervals (written in increasing order) such that

(a) $A_j^{(n)} = A_{2j-1}^{(n+1)} \cup A_{2j}^{(n+1)}$,

(b) $t_n \in A_1^{(n)}$ for infinitely many n,

(c) $\bigcup_{n=1}^\infty \{A_1^{(n)}, A_2^{(n)}, \dots, A_{2^n}^{(n)}\}$ is a basis for the topology on Ω.

Set $\mathcal{F} = \bigcup_{n=1}^\infty \{A_1^{(n)}, A_2^{(n)}, \dots, A_{2^n-1}^{(n)}\}$, and set

$$H_n = \operatorname{span}\{1_{A_j^{(n)}} \mid j = 1, 2, \dots, 2^n - 1\} \subseteq C_0(\Omega_0, \mathbb{Z}).$$

Note that $1_{A_{2^n}^{(n)}}$ does not belong to $C_0(\Omega_0, \mathbb{Z})$ because $1 \in A_{2^n}^{(n)}$.

We outline the idea of the rather lengthy proof below. We show first that $\alpha_n(H_n) \subseteq H_{n+1}$ for all n and that $\bigcup_{n=1}^\infty \alpha_{\infty,n}(H_n) = K_0(\mathcal{A}_\Omega)$, where $\alpha_{\infty,n} = K_0(\varphi_{\infty,n})$ is the inductive limit homomorphism from $C_0(\Omega_0, \mathbb{Z})$ to $K_0(\mathcal{A}_\Omega)$. We then construct positive, injective group homomorphisms $\beta_n \colon H_n \to C_0(\Omega_0, \mathbb{Z}[\frac{1}{2}])$ that satisfy $\beta_{n+1} \circ \alpha_n = \beta_n$ for all n, and which therefore induce a positive injective group homomorphism $\beta \colon K_0(\mathcal{A}_\Omega) \to C_0(\Omega_0, \mathbb{Z}[\frac{1}{2}])$. It is finally proved that β is onto and that $K_0(\mathcal{A}_\Omega)$ is totally ordered, and from this one can conclude that β is an order isomorphism.

For each interval $[r,s] \cap \Omega$ and for each $t \in \Omega$,

$$1_{[r,s]\cap\Omega} \circ \chi_t = \begin{cases} 1_{[r,s]\cap\Omega}, & t < r, \\ 1_{[0,s]\cap\Omega}, & r \le t \le s, \\ 0, & t > s. \end{cases} \tag{6.3}$$

Suppose that $A_1, A_2, \dots, A_m$ is a partition of Ω into clopen intervals, written in increasing order, and that $t \in A_{j_0}$. Then, by (6.3),

$$(6.4)\qquad 1_{A_j} + 1_{A_j} \circ \chi_t = \begin{cases} 1_{A_j}, & j < j_0, \\ 2\cdot 1_{A_j} + 1_{A_{j-1}} + \cdots + 1_{A_1}, & j = j_0, \\ 2\cdot 1_{A_j}, & j > j_0. \end{cases}$$

The lexicographic order on $G = C_0(\Omega_0, \mathbb{Z}[\frac{1}{2}])$ has the following description: If $k \le n$ and if $r_1, r_2, \dots, r_k$ are elements in $\mathbb{Z}[\frac{1}{2}]$ with $r_k \neq 0$, then

$$(6.5)\qquad r_k 1_{A_k} + r_{k-1} 1_{A_{k-1}} + \cdots + r_1 1_{A_1} \in G^+ \iff r_k > 0.$$

It follows from (6.4) that $\alpha_n(H_n) = H_n \subseteq H_{n+1}$. As $\mathcal{F}$ is a basis for the topology of Ω, the set $\{1_A \mid A \in \mathcal{F}\}$ generates $C_0(\Omega_0, \mathbb{Z})$. To prove that $\bigcup_{n=1}^{\infty} \alpha_{\infty,n}(H_n) = K_0(\mathcal{A}_\Omega)$ it suffices to show that $\alpha_{\infty,m}(1_A) \in \bigcup_{n=1}^{\infty} \alpha_{\infty,n}(H_n)$ for every A in $\mathcal{F}$ and for every m in $\mathbb{N}$. Take $A \in \mathcal{F}$ and find a natural number $n \ge m$ such that 1_A belongs to H_n. Let A' be the clopen interval in Ω consisting of all points in Ω that are smaller than $\min A$. Then $1_{A'}$ belongs to H_n, and $\alpha_{n,m}(1_A)$ belongs to $\operatorname{span}\{1_{A'}, 1_A\} \subseteq H_n$ by (6.4). Hence $\alpha_{\infty,m}(1_A) = \alpha_{\infty,n}(\alpha_{n,m}(1_A))$ belongs to $\alpha_{\infty,n}(H_n)$.

The next step is to find a sequence of positive, injective group homomorphisms $\beta_n \colon H_n \to G$ such that $\beta_{n+1} \circ \alpha_n = \beta_n$. (This sequence will then induce a positive, injective group homomorphism $\beta \colon K_0(\mathcal{A}_\Omega) \to G$.) Each function $\{1_{A_1^{(n)}}, 1_{A_2^{(n)}}, \dots, 1_{A_{2^n-1}^{(n)}}\} \to G^+$ extends uniquely to a positive group homomorphism $H_n \to G$, and so it suffices to specify β_n on this generating set. We do so by setting

$$(6.6)\qquad \beta_n(1_{A_j^{(n)}}) = \delta(j,j,n) 1_{A_j^{(n)}} + \sum_{i=1}^{j-1} \delta(j,i,n) 1_{A_i^{(n)}}, \quad j = 1, 2, \dots, 2^n - 1,$$

for suitable coefficients, $\delta(j,i,n)$, in $\mathbb{Z}[\frac{1}{2}]$—to be constructed—such that $\delta(j,j,n) = 2^{-k} > 0$ for some $k \in \mathbb{N}$, and such that $1_{A_j^{(n)}}$ belongs to the image of β_n for $j = 1, 2, \dots, 2^n - 1$. Positivity of β_n will follow from (6.5), (6.6), and the fact that $\delta(j,j,n) > 0$.

For $n = 1$ set $\beta_1(1_{A_1^{(1)}}) = 1_{A_1^{(1)}}$, so that $\delta(1,1,1) = 1$. Suppose that β_n has been found. The point t_n belongs to $A_{j_0}^{(n)}$ for some j_0. The equation $\beta_{n+1}(\alpha_n(1_{A_j^{(n)}})) = \beta_n(1_{A_j^{(n)}})$ has by (6.4) the solution:

$$(6.7)\qquad \beta_{n+1}(1_{A_j^{(n)}}) = \begin{cases} \beta_n(1_{A_j^{(n)}}), & j < j_0, \\ \frac{1}{2}\beta_n(1_{A_j^{(n)}}) - \frac{1}{2}\sum_{i=1}^{j-1} \beta_n(1_{A_i^{(n)}}), & j = j_0, \\ \frac{1}{2}\beta_n(1_{A_j^{(n)}}), & j > j_0. \end{cases}$$

Extend β_{n+1} from H_n to H_{n+1} as follows:

$$\beta_{n+1}(1_{A_{2j-1}^{(n+1)}}) = \delta(j,j,n)1_{A_{2j-1}^{(n+1)}} + \sum_{i=1}^{j-1}\delta(j,i,n)1_{A_i^{(n)}}, \quad j = 1,\dots,j_0-1,$$

$$\beta_{n+1}(1_{A_{2j}^{(n+1)}}) = \delta(j,j,n)1_{A_{2j}^{(n+1)}}, \quad j = 1,\dots,j_0-1,$$

$$\beta_{n+1}(1_{A_{2j-1}^{(n+1)}}) = \frac{1}{2}\delta(j,j,n)1_{A_{2j-1}^{(n+1)}} + \frac{1}{2}\sum_{i=1}^{j-1}\Big(\delta(j,i,n) - \sum_{k=i}^{j-1}\delta(k,i,n)\Big)1_{A_i^{(n)}}, \quad j = j_0,$$

$$\beta_{n+1}(1_{A_{2j}^{(n+1)}}) = \frac{1}{2}\delta(j,j,n)1_{A_{2j}^{(n+1)}}, \quad j = j_0,$$

$$\beta_{n+1}(1_{A_{2j-1}^{(n+1)}}) = \frac{1}{2}\delta(j,j,n)1_{A_{2j-1}^{(n+1)}} + \frac{1}{2}\sum_{i=1}^{j-1}\delta(j,i,n)1_{A_i^{(n)}}, \quad j = j_0+1,\dots,2^n-1,$$

$$\beta_{n+1}(1_{A_{2j}^{(n+1)}}) = \frac{1}{2}\delta(j,j,n)1_{A_{2j}^{(n+1)}}, \quad j = j_0+1,\dots,2^n-1,$$

$$\beta_{n+1}(1_{A_{2^n-1}^{(n+1)}}) = 1_{A_{2^n-1}^{(n+1)}}.$$

The coefficients, that appear implicit in these expressions for $\beta_{n+1}(1_{A_j^{(n+1)}})$, will be our $\delta(j,i,n+1)$.

It follows by induction on n that each coefficient $\delta(j,i,n)$ belongs to $\mathbb{Z}[\frac{1}{2}]$ and that $\delta(j,j,n) = 2^{-k}$ for some $k \in \mathbb{N}$ (that depends on j and n). The formula above for β_{n+1} is consistent with (6.7), and so $\beta_{n+1} \circ \alpha_n = \beta_n$. It also follows by induction on n that $1_{A_j^{(n)}}$ belongs to $\mathrm{Im}(\beta_n)$ for $j = 1, 2, \dots, 2^n - 1$. This clearly holds for $n = 1$. Assume it holds for some $n \geq 1$. Then $1_{A_j^{(n)}}$ belongs to $\mathrm{Im}(\beta_n) \subseteq \mathrm{Im}(\beta_{n+1})$ for $j = 1, 2, \dots, 2^n - 1$, and hence $1_{A_{2j}^{(n+1)}}$, $1_{A_{2j-1}^{(n+1)}} = 1_{A_j^{(n)}} - 1_{A_{2j}^{(n+1)}}$, and $1_{A_{2^n-1}^{(n+1)}}$ belongs to $\mathrm{Im}(\beta_{n+1})$. It is now verified that each β_n is as desired.

To complete the proof we must show that the positive, injective, group homomorphism $\beta\colon K_0(\mathcal{A}_\Omega) \to G$ is surjective and that $\beta(K_0(\mathcal{A}_\Omega)^+) = G^+$. The former follows from the already established fact that 1_A belongs to the image of β for all $A \in \mathcal{F}$, and from the fact, which follows from Proposition 5.2, that if f belongs to $\mathrm{Im}(\beta)$, then so does $\frac{1}{2}f$. The latter identity is proved by verifying that $K_0(\mathcal{A}_\Omega)$ is totally ordered.

To show that $K_0(\mathcal{A}_\Omega)$ is totally ordered we must show that either f or $-f$ is positive for each non-zero f in $K_0(\mathcal{A}_\Omega)$. Write $f = \alpha_{\infty,n}(g)$ for a suitable n and $g \in C_0(\Omega_0, \mathbb{Z})$. Let r be the largest point in Ω for which $g(r) \neq 0$. Upon replacing f by $-f$, if necessary, we can assume that $g(r)$ is positive. There is a (non-empty) clopen interval $A = [s,r] \cap \Omega$ for which $g(t) \geq 1$ for all t in A. Put $X_{k,n} = \{t_n, t_{n+1}, \ldots, t_{n+k-1}\}$, $Y_{k,n} = X_{k,n} \cap [0,r]$, and $Z_{k,n} = X_{k,n} \cap [0,s)$. By (6.3) and an analog of (2.3) we get

$$\begin{aligned}\alpha_{n+k,n}(g) &= \sum_{F \subseteq X_{k,n}} g \circ \chi_{\max F} = \sum_{F \subseteq Y_{k,n}} g \circ \chi_{\max F} \\ &\geq \sum_{F \subseteq Z_{k,n}} \min g(\Omega_0) + \sum_{F \subseteq Y_{k,n}, F \not\subseteq Z_{k,n}} 1_A \circ \chi_{\max F} \\ &= 2^{|Z_{k,n}|} \cdot \min g(\Omega_0) + (2^{|Y_{k,n}|} - 2^{|Z_{k,n}|}) \cdot 1_{[0,r] \cap \Omega}.\end{aligned}$$

Now,

$$\lim_{k \to \infty} (|Y_{k,n}| - |Z_{k,n}|) = \lim_{k \to \infty} |X_{k,n} \cap [r,s]| = \infty,$$

so $\alpha_{n+k,n}(g) \geq 0$ for some large enough k. But then $f = \alpha_{\infty,n+k}(\alpha_{n+k,n}(g))$ is positive. ∎

References

[1] B. Blackadar, *Matricial and ultra matricial topology*, in *Operator Algebras, Mathematical Physics, and Low-dimensional Topology (Istanbul, 1991)* (A. K. Peters, ed.), Research Notes in Mathematics, Vol. 5, Wellesley, Mass., 1993, pp. 11–38.

[2] B. Blackadar, A. Kumjian and M. Rørdam, *Approximately central matrix units and the structure of non-commutative tori*, K-theory **6** (1992), 267–284.

[3] E. Blanchard and E. Kirchberg, *Non simple purely infinite C^*-algebras: The Hausdorff case*, Journal of Functional Analysis, to appear.

[4] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, Journal of Functional Analysis **99** (1991), 131–149.

[5] G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, Journal of Algebra **38** (1976), 29–44.

[6] K. R. Goodearl, *Notes on a class of simple C^*-algebras with real rank zero*, Publicacions Matemàtiques **36** (1992), 637–654.

[7] U. Haagerup, *Every quasi-trace on an exact C^*-algebra is a trace*, preprint, 1991.

[8] J. Hjelmborg and M. Rørdam, *On stability of C^*-algebras*, Journal of Functional Analysis **155** (1998), 153–170.

[9] E. Kirchberg, *The classification of Purely Infinite C^*-algebras using Kasparov's Theory*, in preparation.

[10] E. Kirchberg, *On the existence of traces on exact stably projectionless simple C^*-algebras*, in *Operator Algebras and their Applications* (P. A. Fillmore and J. A. Mingo, eds.), Fields Institute Communications, Vol. 13, American Mathematical Society, Providence, RI, 1995, pp. 171–172.

[11] E. Kirchberg and N. C. Phillips, *Embedding of exact C^*-algebras into $\mathcal{O}_2$*, Journal für die reine und angewandte Mathematik **525** (2000), 17–53.

[12] E. Kirchberg and M. Rørdam, *Non-simple purely infinite C^*-algebras*, American Journal of Mathematics **122** (2000), 637–666.

[13] E. Kirchberg and M. Rørdam, *Infinite non-simple C^*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$*, Advances in Mathematics **167** (2002), 195–264.

[14] E. Kirchberg and M. Rørdam, *Purely infinite C^*-algebras: Ideal preserving zero homotopies*, Preprint, 2003.

[15] J. Mortensen, *Classification of certain non-simple C^*-algebras*, Journal of Operator Theory **41** (1999), 223–259.

[16] N. Ozawa, *Homotopy invariance of AF-embeddability*, Geometric and Functional Analysis **39** (2003), 216–222.

[17] M. Rørdam, *A simple C^*-algebra with a finite and an infinite projection*, Acta Mathematica **191** (2003), 109–142.

[18] M. Rørdam, F. Larsen and N. J. Laustsen, *An Introduction to K-theory for C^*-algebras*, London Mathematical Society — Student Texts, Vol. 49, Cambridge University Press, Cambridge, 2000.

[19] D. Voiculescu, *A note on quasidiagonal C^*-algebras and homotopy*, Duke Mathematical Journal **62** (1991), 267–271.